The Drude Theory of Metals - continued

The density of “electron gas” >> classic gas

Drude Model Assumptions:

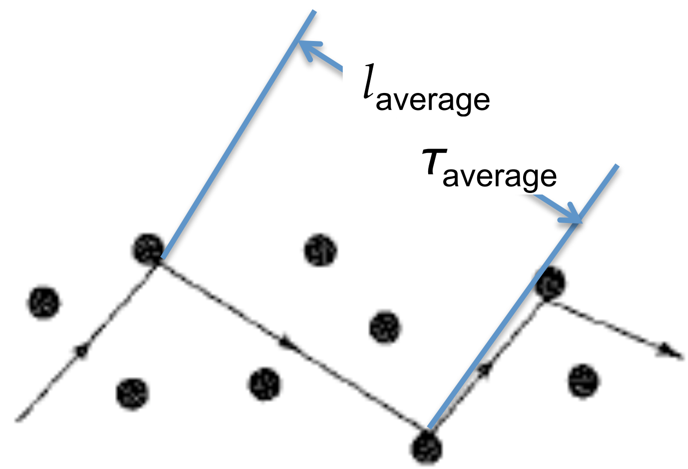
- Travel in straight line, ignore e-e interactions (i.e., *independent electron approximation*)
- Ignore electron-ion interactions (i.e., *free electron approximation*)
- Collision instantaneously alters the velocity of an electron
- The relaxation time (*mean free time*) is independent of electron position and velocity
- Achieve thermal equilibrium only through collision

DC electrical conductivity: $\sigma = \dfrac{ne^2\boldsymbol{\tau}}{m}$

Mean free path vs. mean free time (relaxation time)

$$\sigma = \frac{ne^2\tau}{m}$$

$$\tau = \frac{m\sigma}{ne^2} = \frac{m}{\rho ne^2}$$ $10^{-14} - 10^{-15}$ sec



$$l_{ave} = v_{ave}\tau_{ave}$$

estimate $v_{ave} \sim 10^7 cm/\mathbf{sec}$ assuming $\frac{1}{2}mv_{ave}^2 = \frac{3}{2}k_BT$ (it is incorrect!)

Mean free path: l_{ave} : 1 to 10 Å

Warning: this estimate is oversimplified

DRUDE RELAXATION TIMES IN UNITS OF 10^{-14} SECOND[a]

ELEMENT	77 K	273 K	373 K
Li	7.3	0.88	0.61
Na	17	3.2	
K	18	4.1	
Rb	14	2.8	
Cs	8.6	2.1	
Cu	21	2.7	1.9
Ag	20	4.0	2.8
Au	12	3.0	2.1
Be		0.51	0.27
Mg	6.7	1.1	0.74
Ca		2.2	1.5
Sr	1.4	0.44	
Ba	0.66	0.19	
Nb	2.1	0.42	0.33
Fe	3.2	0.24	0.14
Zn	2.4	0.49	0.34
Cd	2.4	0.56	
Hg	0.71		
Al	6.5	0.80	0.55
Ga	0.84	0.17	
In	1.7	0.38	0.25
Tl	0.91	0.22	0.15
Sn	1.1	0.23	0.15
Pb	0.57	0.14	0.099
Bi	0.072	0.023	0.016
Sb	0.27	0.055	0.036

Equation of Motion

$$t \rightarrow t + dt$$

Collision Probability: dt/τ

Non-collision Probability: 1-dt/τ

Force due to either electrical field or Magnetic field

Their momentum: $\mathbf{p}(t) + \mathbf{f}(t)dt$

$t + dt$

$$\mathbf{p}(t + dt) = \left(1 - \frac{dt}{\tau}\right)\left(\mathbf{p}(t) + \mathbf{f}(t)dt\right)$$

$$\frac{d\,\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t)$$

$$\frac{d\,\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t)$$

-- The effect of individual electron collision is to introduce a damping term, **f**(t) into the equation of motion for the momentum

AC Electrical Conductivity σ(ω)

$$\mathbf{E}(t) \Rightarrow \mathbf{E}(\omega)e^{-i\omega t} \qquad \mathbf{p}(t) \Rightarrow \mathbf{p}(\omega)e^{-i\omega t} \qquad \mathbf{f}(t) = -e\,\mathbf{E}(t)$$

$$\frac{d\,\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t)$$

$$\mathbf{J}(\omega) = -ne\,\mathbf{v}(\omega) = -\frac{ne\,\mathbf{p}(\omega)}{m} = \frac{\left(ne^2/m\right)}{\left(1/\tau\right) - i\omega}\mathbf{E}(\omega)$$

$$\mathbf{J}(\omega) = \sigma(\omega)\,\mathbf{E}(\omega) \longrightarrow \sigma(\omega) = \frac{\left(ne^2\tau/m\right)}{1 - i\omega\tau}$$

$$\sigma(\omega) = \frac{\left(ne^2\tau/m\right)}{1 - i\omega\tau}$$

- Ignored magnetic field **H** generated by **E**
- **E** is spatially uniform

Considering these two, we need to use Maxwell's equations

$$\nabla \bullet \mathbf{E} = 0 \qquad \nabla \bullet \mathbf{H} = 0 \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \qquad \Rightarrow -\nabla^2 \mathbf{E} = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi\sigma}{\omega} i\right)\mathbf{E}$$

Complex dielectric constant: $\varepsilon(\omega) = 1 + \frac{4\pi\sigma}{\omega} i$

$$\sigma(\omega) = \frac{\left(ne^2\tau/m\right)}{1 - i\omega\tau} \qquad \varepsilon(\omega) = 1 + \frac{4\pi\sigma}{\omega} i$$

$$\omega\tau >> 1$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

Plasma frequency: $\omega_{plasma}^2 = \frac{4\pi ne^2}{m}$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\varepsilon(\omega) < 0 \qquad \text{if} \qquad \omega < \omega_p$$

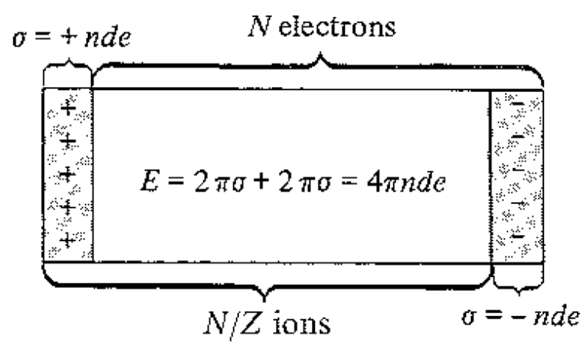
-- the system is too conductive.
No radiation can propagate

$$\varepsilon(\omega) > 0 \qquad \text{if} \qquad \omega > \omega_p$$

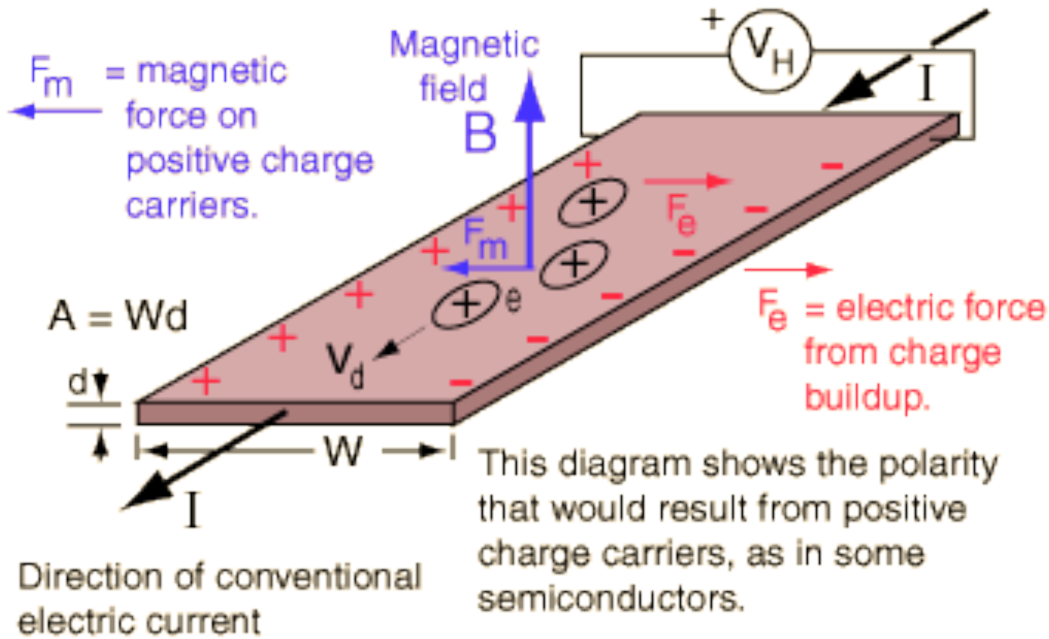
-- Radiation can propagate

$$\varepsilon(\omega) = 0 \qquad \text{if} \qquad \omega = \omega_p \quad \left(1 + \frac{4\pi i \sigma(\omega)}{\omega} = 0\right)$$

-- Plasma oscillation (plasmon)

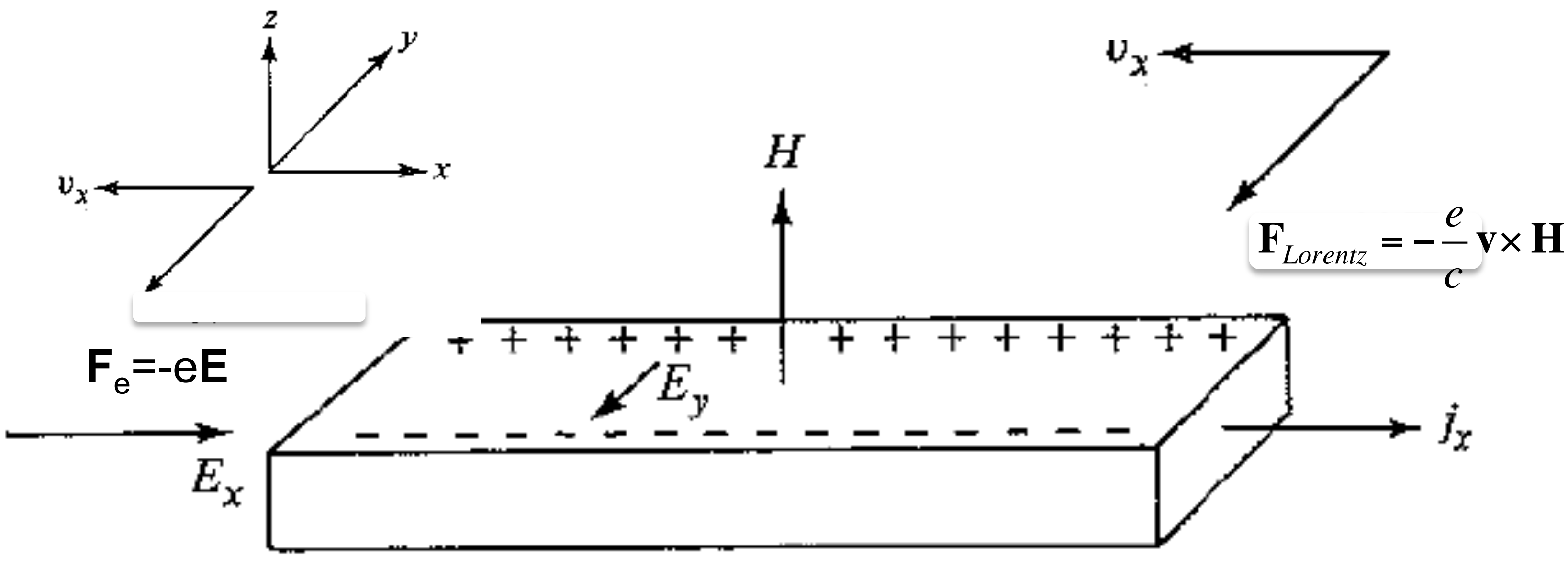


Drude Confirms Hall Effect



Drude's Approach

$$\frac{d\,\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t)$$



$$\mathbf{f}(t) = -\frac{e}{c}\mathbf{v}\times\mathbf{H} - e\,\mathbf{E}$$

Drude's Approach

$$\frac{d\,\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} + \mathbf{f}(t) \qquad \mathbf{f}(t) = -\frac{e}{c}\mathbf{v}\times\mathbf{H} - e\,\mathbf{E}$$

$$\frac{d\,\mathbf{p}(t)}{dt} = -\frac{\mathbf{p}(t)}{\tau} - \frac{e}{c}\mathbf{v}\times\mathbf{H} - e\,\mathbf{E}$$

Steady state $\quad \frac{d\,\mathbf{p}(t)}{dt} = 0$

$$-\frac{\mathbf{p}(t)}{\tau} - \frac{e}{c}\mathbf{v}\times\mathbf{H} - e\,\mathbf{E} = 0$$

Drude's Approach

$$-\frac{\mathbf{p}(t)}{\tau}-\frac{e}{c}\mathbf{v}\times\mathbf{H}-e\mathbf{E}=0$$

$$\mathbf{v}=\frac{\mathbf{p}}{m}$$

X-direction: $$-eE_x-\left(\frac{eH}{mc}\right)p_y-\frac{p_x}{\tau}=0$$

y-direction: $$-eE_y+\left(\frac{eH}{mc}\right)p_x-\frac{p_y}{\tau}=0$$

$$p_x=-\frac{m}{ne}j_x \qquad p_y=-\frac{m}{ne}j_y$$

$$\sigma_0^{DC}E_x=j_x+\omega_c\tau j_y \qquad \sigma_0^{DC}E_y=-\omega_c\tau j_x+j_y$$

Drude's Approach

$$\sigma_0^{DC} E_x = j_x + \omega_c \tau j_y$$

$$\sigma_0^{DC} E_y = -\omega_c \tau j_x + j_y$$

$$\omega_c = \frac{eH}{mc}$$

(cyclotron frequency)

$$\downarrow \; j_y = 0$$

Longitudinal resistivity (x-direction): $\rho_x = \frac{E_x}{j_x} = \sigma_0^{DC}$

Transverse resistivity (y-direction)
=Hall resistivity: $\rho_{xy} = \frac{E_y}{j_x} = -\frac{H}{nec}$

Hall coefficient: $R_H = \frac{\rho_{xy}}{H} = -\frac{1}{nec}$

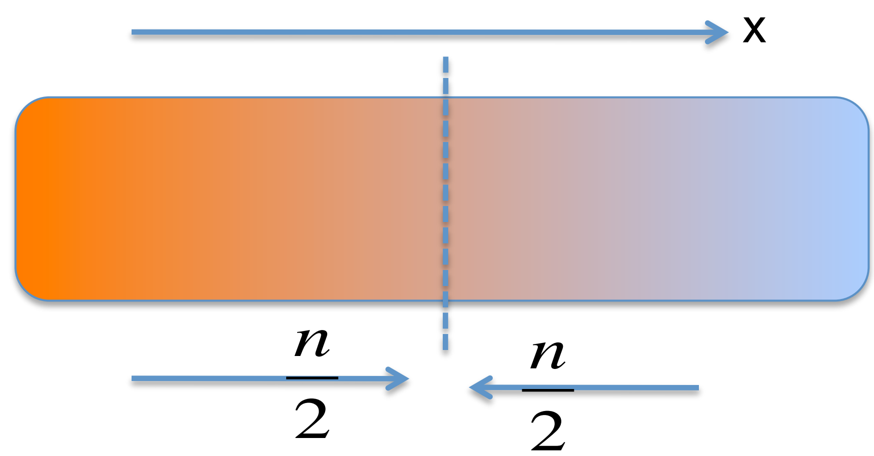
Drude Confirms Empirical Law of Wiedemann + Franz

$$\frac{\kappa}{\sigma T} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 = \text{constant}$$

$$\kappa = thermal - conductivity$$

Drude's Approach

Assumption: heat is carried by the conduction electrons



Number of carriers

Thermal current $\frac{n}{2} v_x \varepsilon\left[T\left(x - v_x \tau\right)\right] \; - \frac{n}{2} v_x \varepsilon\left[T\left(x + v_x \tau\right)\right]$

Total thermal current $j_x^Q = \frac{n}{2} v_x \left\{ \varepsilon\left[T\left(x - v_x \tau\right)\right] - \varepsilon\left[T\left(x + v_x \tau\right)\right] \right\}$

$$= n v_x^2 \tau \frac{d\varepsilon}{dT}\left(-\frac{dT}{dx}\right)$$

Drude's Approach

Total thermal current in x-direction $j_x^Q = n v_x^2 \tau \frac{d\varepsilon}{dT}\left(-\frac{dT}{dx}\right)$

Total thermal current in all directions: $\mathbf{j}^Q = \left(\frac{1}{3}v^2\right)\tau\left(n\frac{d\varepsilon}{dT}\right)(-\nabla \mathbf{T})$

$$c_v = n\frac{d\varepsilon}{dT} \Rightarrow \mathbf{j}^Q = \left(\frac{1}{3}v^2\right)\tau c_v(-\nabla \mathbf{T})$$

$$\mathbf{j}^Q = \kappa(-\nabla \mathbf{T})$$

Thermal conductivity: $\kappa = \frac{1}{3}v^2\tau c_v$

Drude's Approach

Thermal conductivity: $$\kappa = \frac{1}{3} v^2 \tau c_v$$

$$\frac{\kappa}{\sigma T} = \frac{\frac{1}{3} v^2 \tau c_v}{\left(\frac{ne^2\tau}{m}\right) T} = \frac{3}{2}\left(\frac{k_B}{e}\right)^2$$

$c_v = \frac{3}{2} n k_B$ $\quad$ $\frac{1}{2} m v^2 = \frac{3}{2} k_B T$ were used

Correct Answer:

$$\frac{\kappa}{\sigma T} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 = 2.44 \times 10^{-8}\, W\Omega K^{-2} = Lorenz - number$$

Drude's calculation had incorrect c_v and 1/2mv^2:

$$c_v \neq \frac{3}{2} n k_B \qquad \frac{1}{2} m v^2 \neq \frac{3}{2} k_B T$$