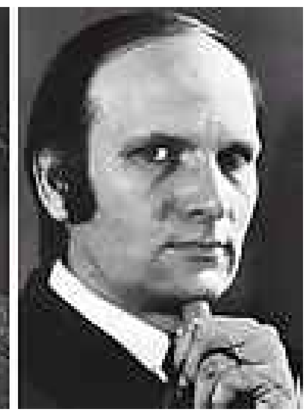
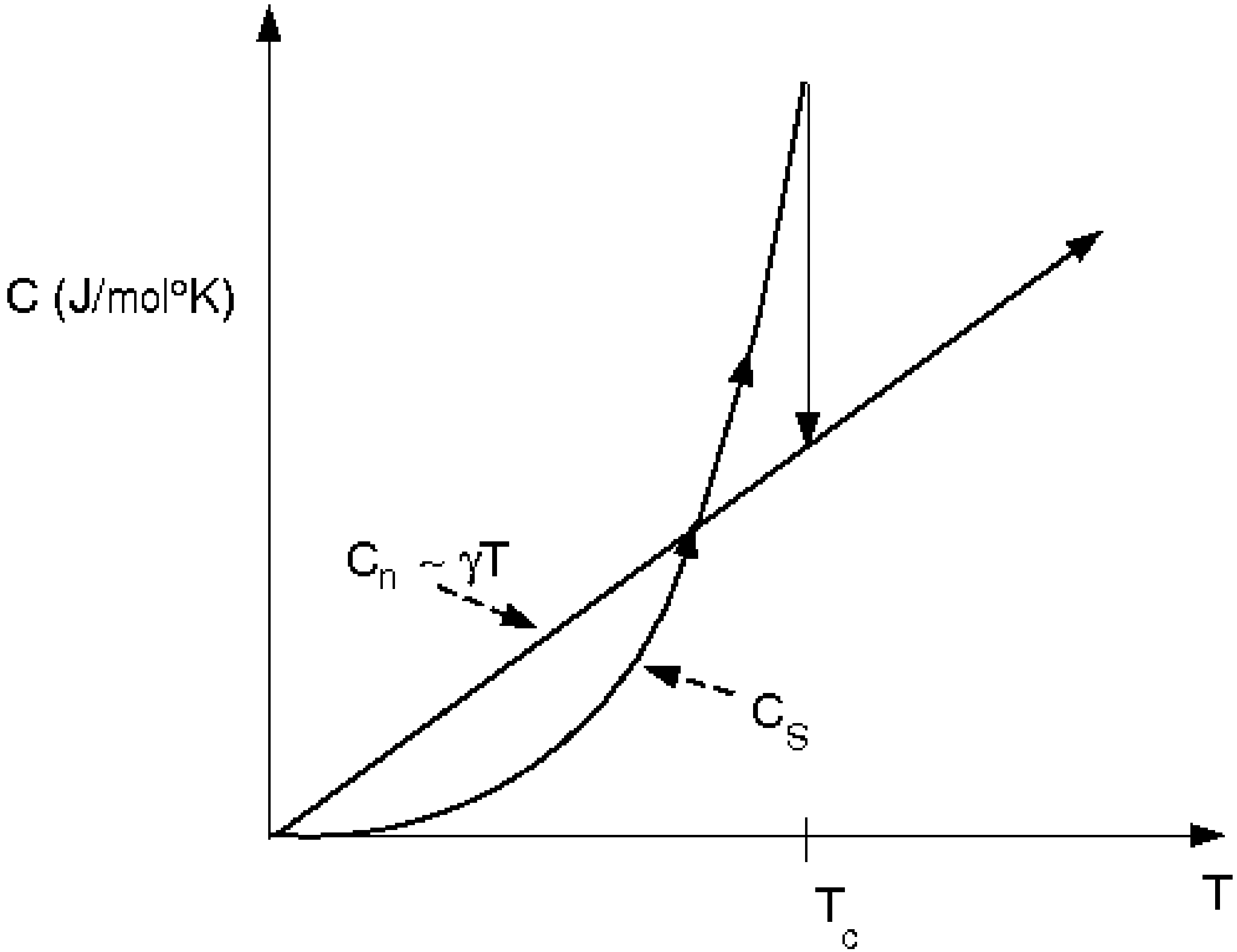
Superconductivity





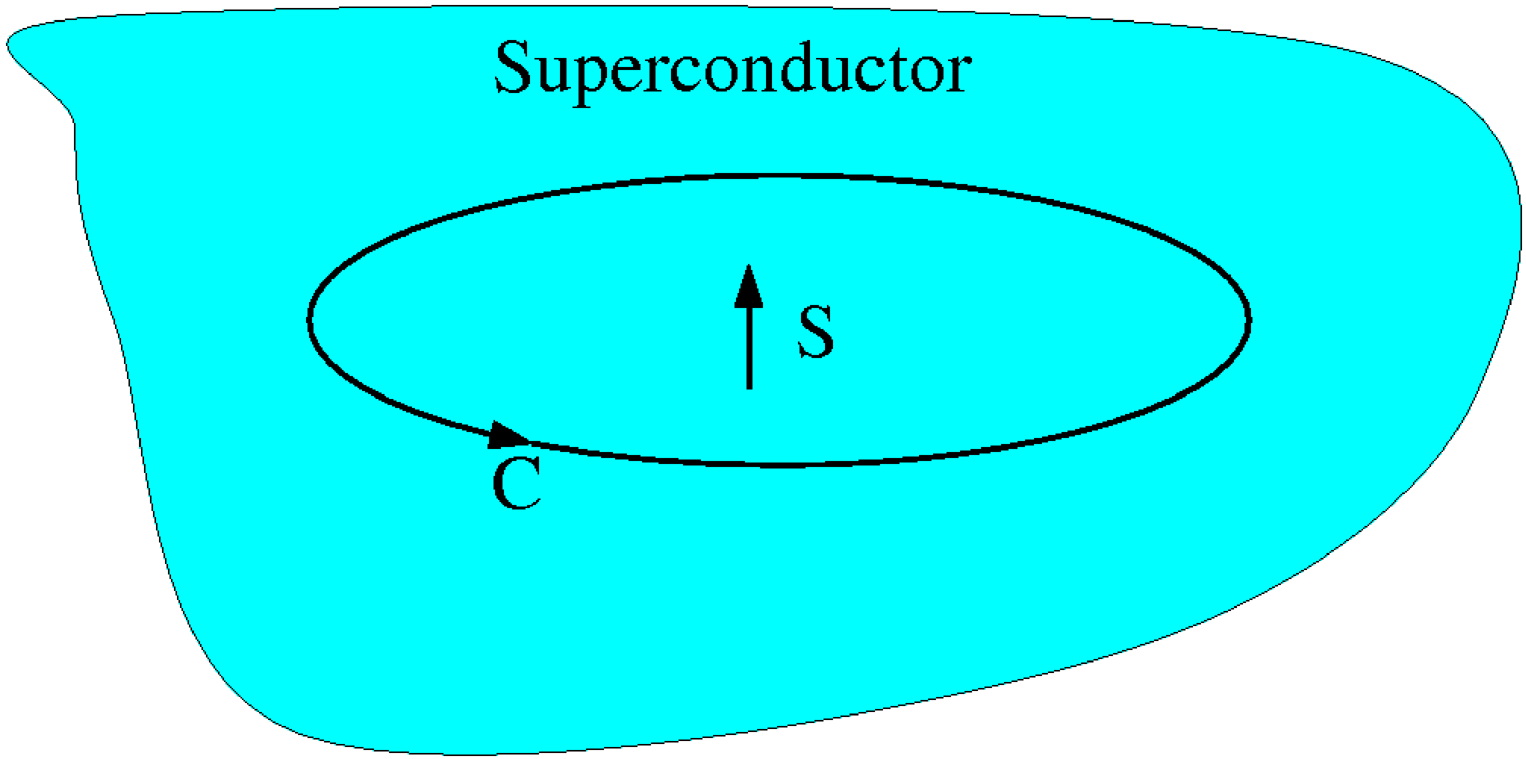
John Bardeen | Leon Cooper | John Schrieffer

Evidence of a Phase Transition



$$C_s \sim e^{-\beta\Delta}$$

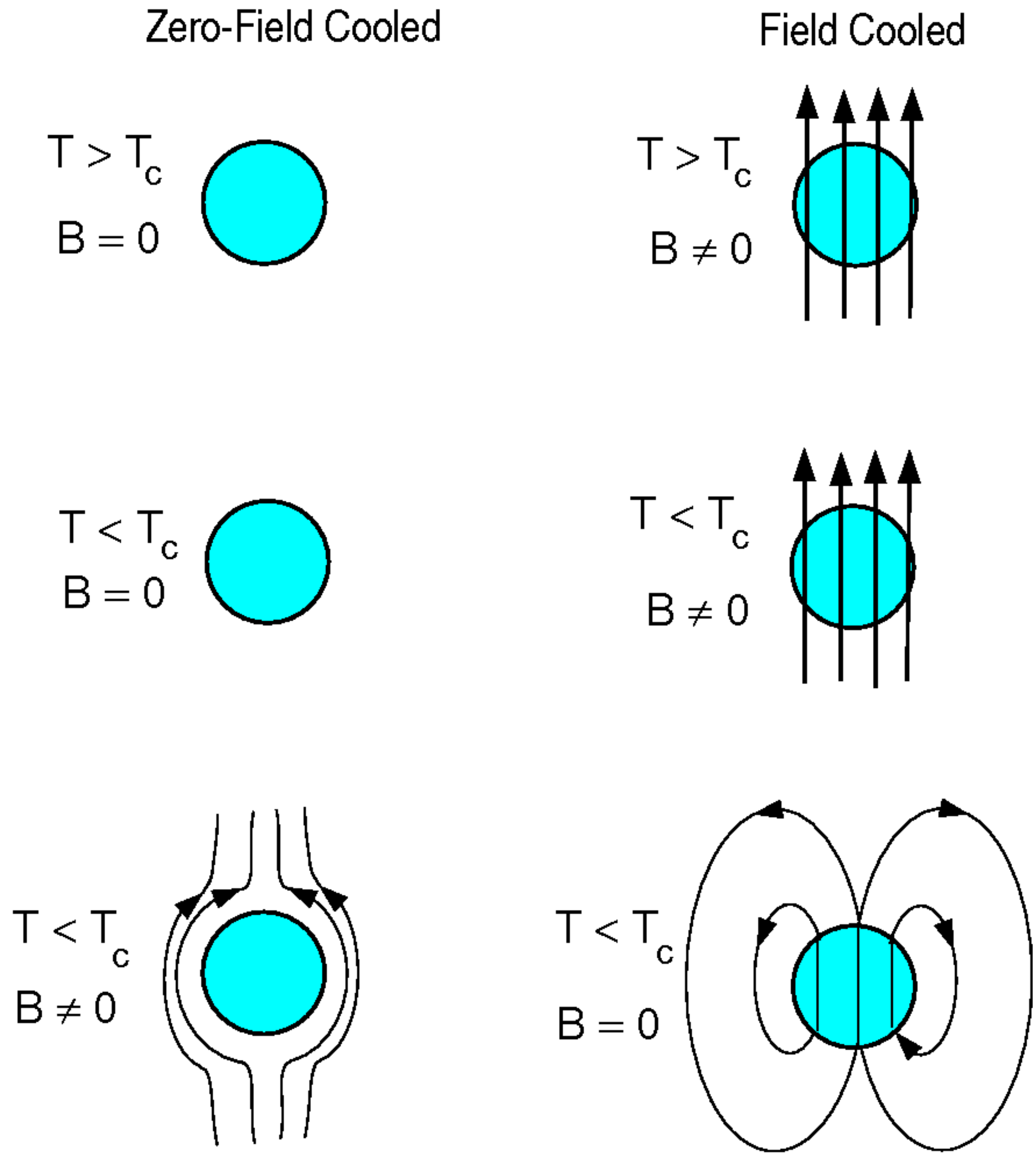
Meissner Effect



$$0 = IR = V = \oint \mathcal{E} \cdot d\mathsf{l} = \int_S \nabla \times \mathcal{E} \cdot d\mathsf{S} = -\frac{1}{c} \int_S \frac{\partial \mathsf{B}}{\partial t} \cdot d\mathsf{S}$$

or, since S and C are arbitrary $0 = -\frac{1}{c}\dot{\mathsf{B}} \cdot S \Rightarrow \dot{\mathsf{B}} = 0$

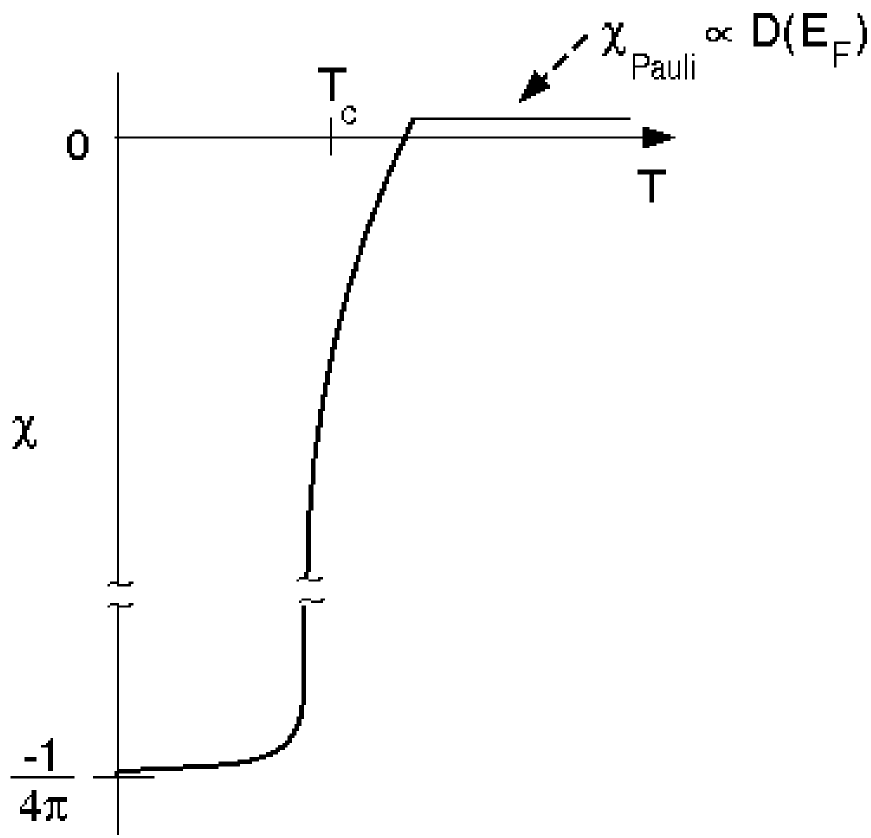
Ideal Conductor

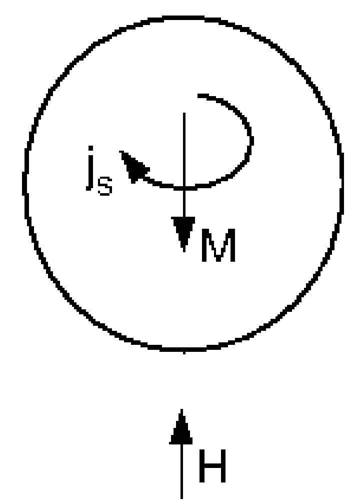


A superconductor is an ideal diamagnet. I.e.

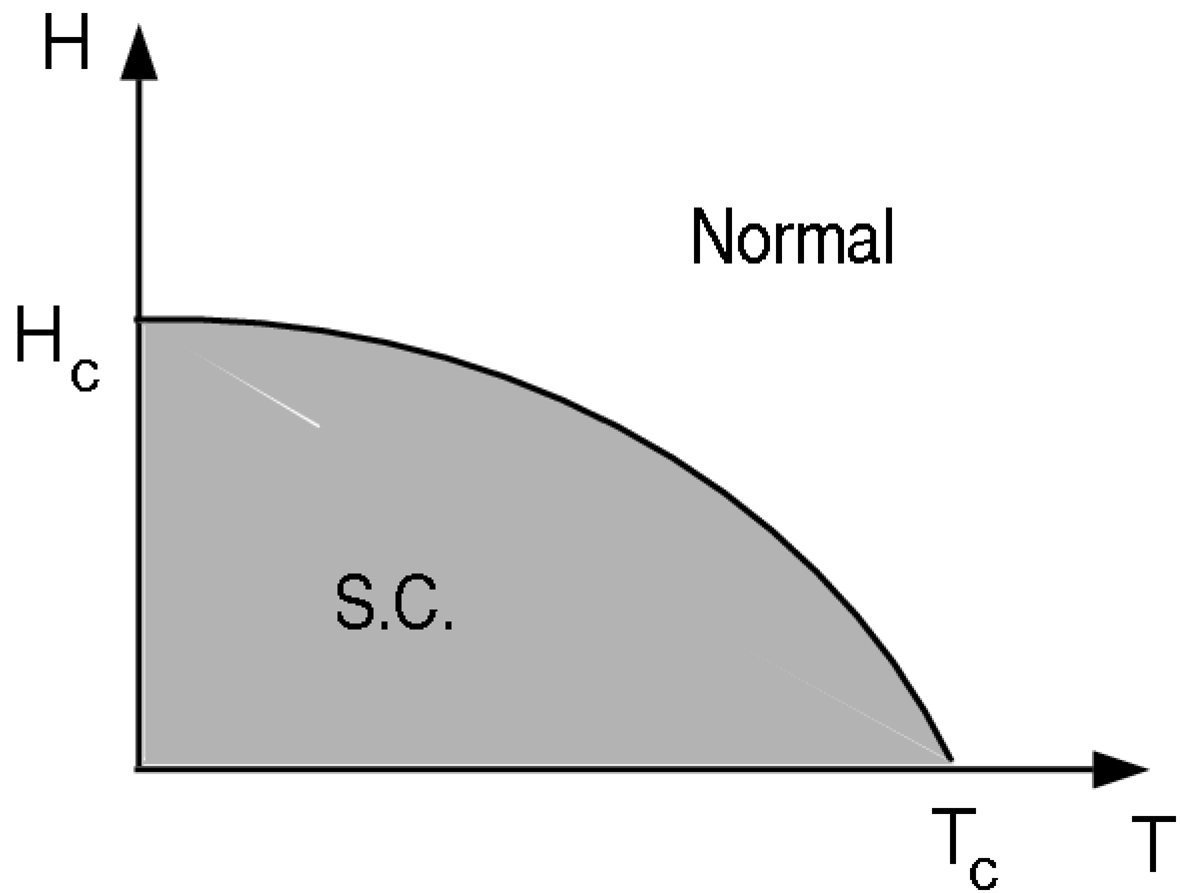
$$\mathbf{B} = \mu H = 0 \Rightarrow \mu = 0 \qquad \mathbf{M} = \chi H = \frac{\mu - 1}{4\pi} H$$

$$\chi_{SC} = -\frac{1}{4\pi}$$





$$M = -\frac{1}{4\pi} H_{\text{ext}}$$



The London Equations

$$m\dot{\mathsf{v}}d = -e\mathcal{E}$$

or, since $\partial\mathbf{j}/\partial t = -en_s\dot{\mathbf{v}}$,

$$\frac{\partial \mathrm{j}_s}{\partial t} = \frac{e^2 n_s}{m}\mathcal{E} \qquad \text{(First London Eqn.)}$$

Then, using the Maxwell equation

$$\nabla \times \mathcal{E} = -\frac{1}{c}\frac{\partial \mathsf{B}}{\partial t} \Rightarrow \frac{m}{n_s e^2}\nabla \times \frac{\partial \mathsf{j}_s}{\partial t} + \frac{1}{c}\frac{\partial \mathsf{B}}{\partial t} = 0$$

or

$$\frac{\partial}{\partial t}\left(\frac{m}{n_s e^2}\nabla \times \mathsf{j}_s + \frac{1}{c}\mathsf{B}\right) = 0$$

$$\nabla \times \mathrm{j}_s = -\frac{n_s e^2}{mc}\mathsf{B} \qquad \text{(Second London Eqn.)}$$

or defining $m/n_s e^2$ the London Equations become

$$\frac{\mathsf{B}}{c} = -\lambda_L \nabla \times \mathsf{j}_s \qquad \mathcal{E} = \lambda_L \frac{\partial \mathsf{j}_s}{\partial t}$$

If we now apply the Maxwell equation $\nabla\times H = 4\pi/c\ j$
$\Rightarrow \nabla\times B = 4\pi/c\ \mu j$ then we get

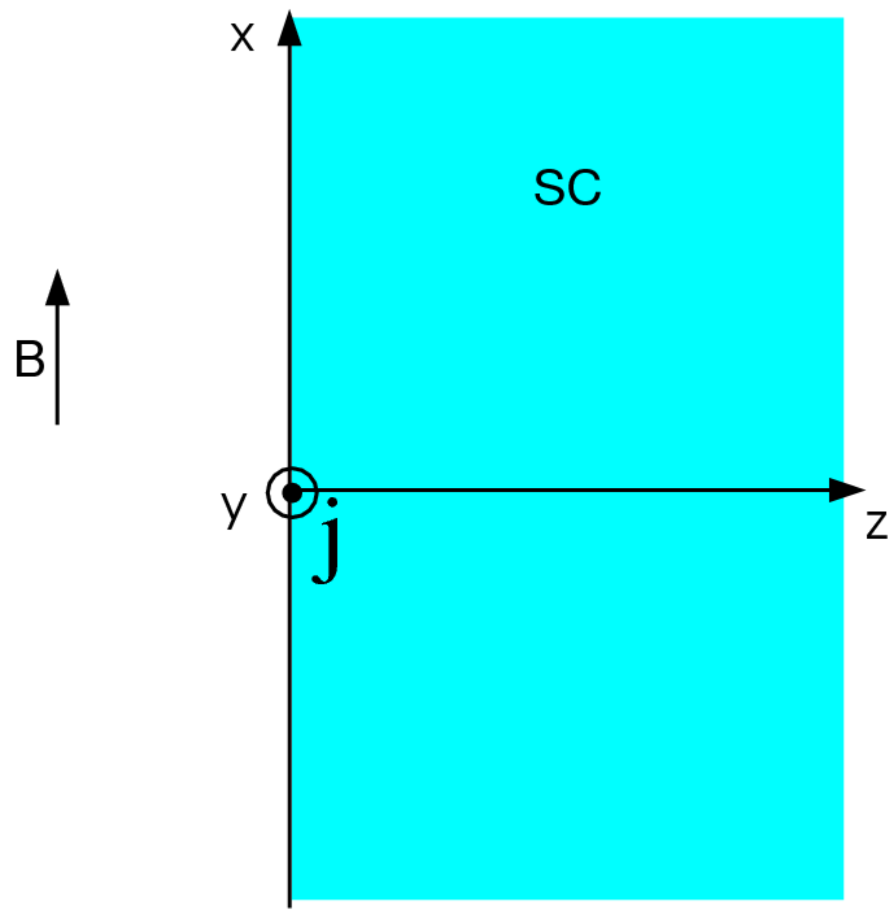
$$\nabla \times (\nabla \times \mathsf{B}) = \frac{4\pi}{c}\mu\nabla \times \mathsf{j} = -\frac{4\pi\mu}{c^2\lambda_L}\mathsf{B}$$

and

$$\nabla \times (\nabla \times \mathsf{j}) = -\frac{1}{\lambda_L c}\nabla \times \mathsf{B} = -\frac{4\pi\mu}{c^2\lambda_L}\mathsf{j}$$

or since $\nabla \cdot \mathsf{B} = 0$, $\nabla \cdot \mathsf{j} = \frac{1}{c}\frac{\partial \rho}{\partial t} = 0$ and
$\nabla \times (\nabla \times \mathsf{a}) = \nabla(\nabla \cdot \mathsf{a}) - \nabla^2 \mathsf{a}$ we get

$$\nabla^2 \mathsf{B} - \frac{4\pi\mu}{c^2 \lambda_L} \mathsf{B} = 0 \qquad \nabla^2 \mathsf{j} - \frac{4\pi\mu}{c^2 \lambda_L} \mathsf{j} = 0$$



$$j_s \propto \nabla \times B \propto \hat{z} \times \hat{x} \ \frac{\partial B_x}{\partial x}$$

Now consider a the superconductor in an external field. The field is only in the x-direction, and can vary in space only in the z-direction, then since ∇×B=4π/c μj, the current is in the y-direction, so

$$\frac{\partial^2 \mathsf{B}_x}{\partial z^2} - \frac{4\pi\mu}{c^2\lambda_L}\mathsf{B}_x = 0 \qquad \frac{\partial^2 \mathsf{j}_{sy}}{\partial z^2} - \frac{4\pi\mu}{c^2\lambda_L}\mathsf{j}_{sy} = 0$$

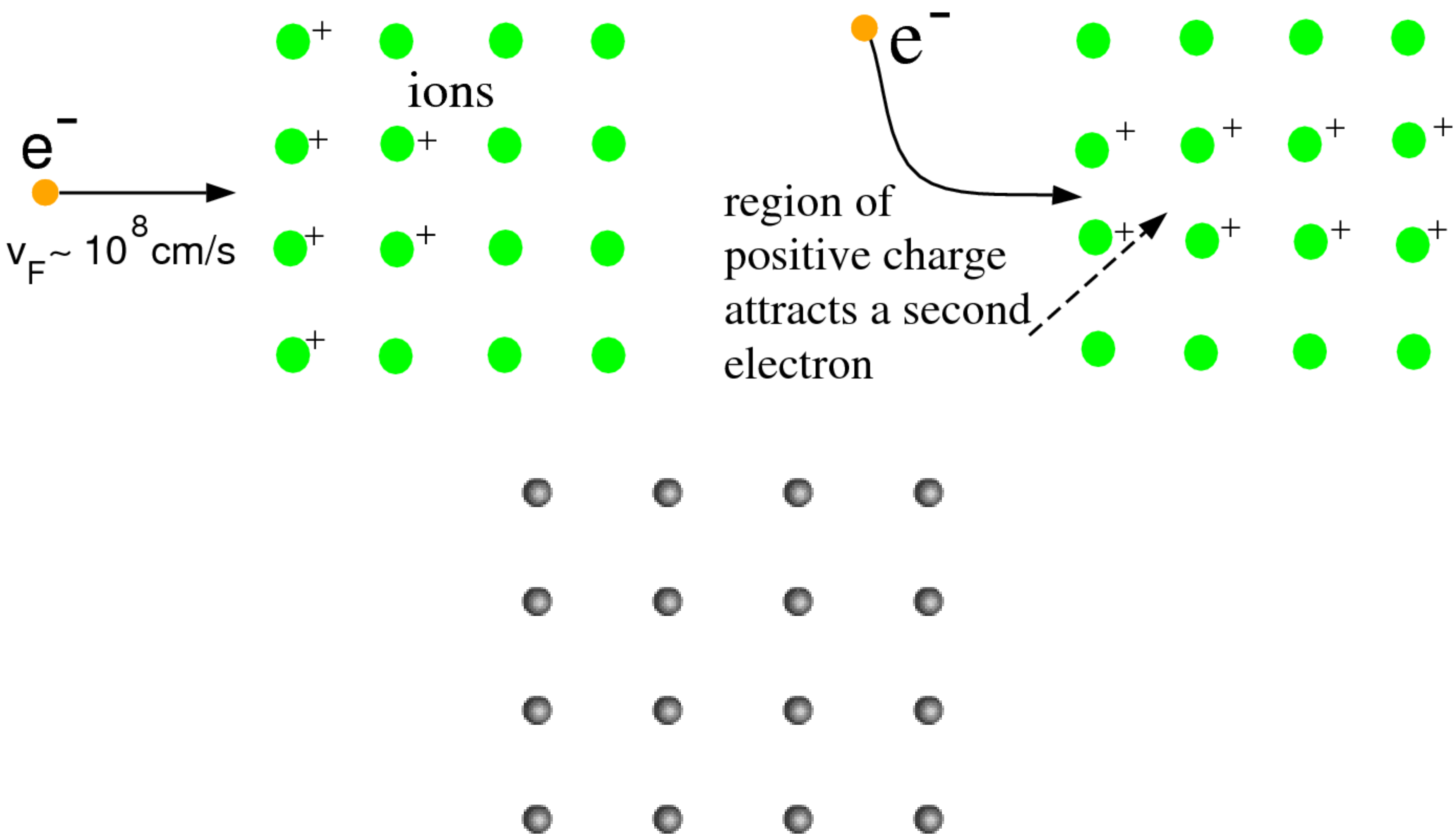
with the solutions

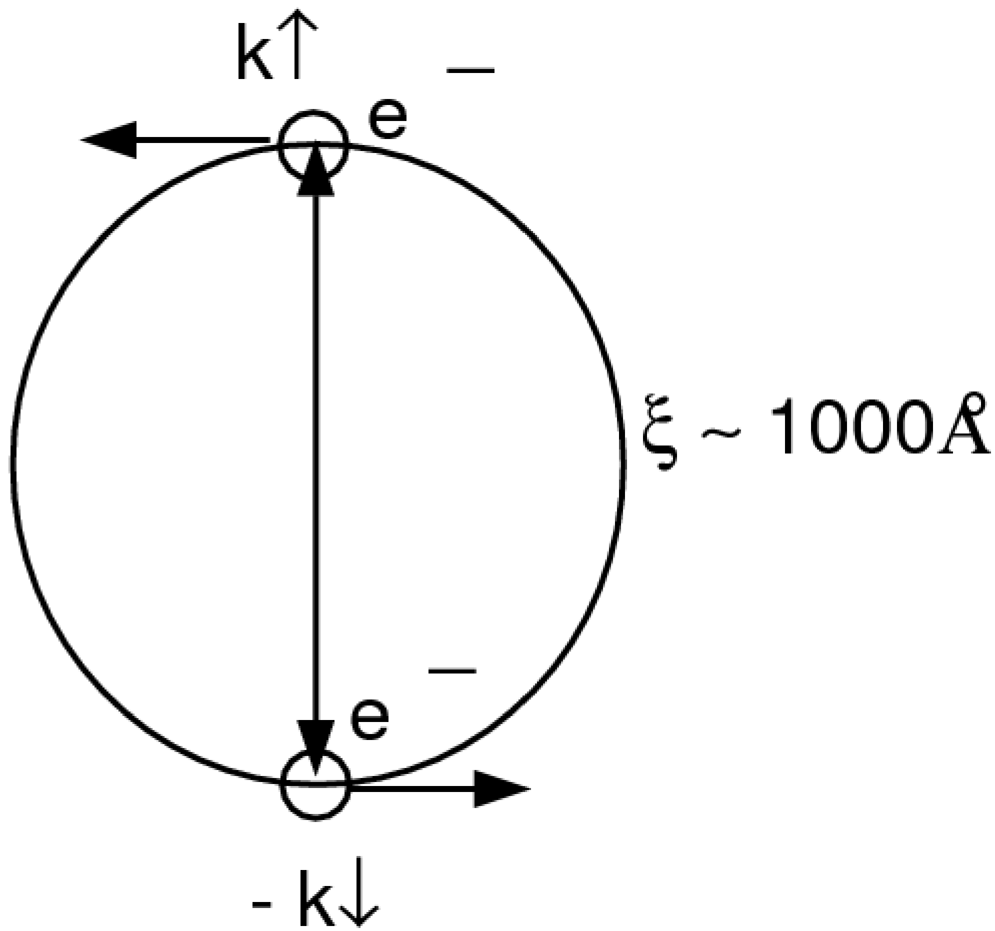
$$\mathsf{B}_x = \mathsf{B}_x^0 e^{-\frac{z}{\Lambda_L}} \qquad \mathsf{j}_{sy} = \mathsf{j}_{sy} e^{-\frac{z}{\Lambda_L}}$$

wher $\Lambda_L = \sqrt{\frac{c^2\lambda_L}{4\pi\mu}} = \sqrt{\frac{mc^2}{4\pi n e^2 \mu}}$ is the penetration depth.

Cooper Pairing

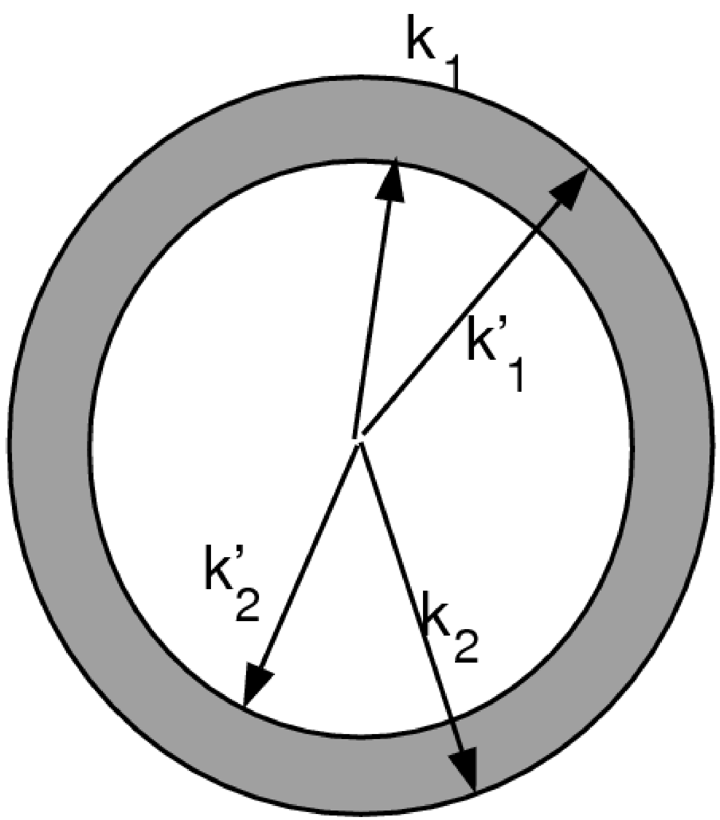
The Retarded Pairing Potential



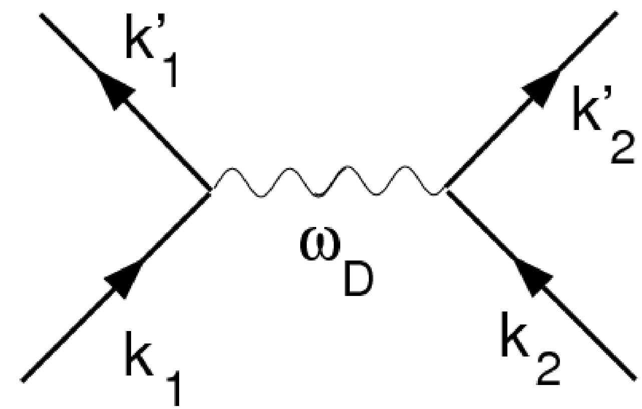


To take full advantage of the attractive potential, the spatial part of the electronic pair wave function is symmetric and hence nodeless. To obey the Pauli principle, the spin part must then be antisymmetric or a singlet.

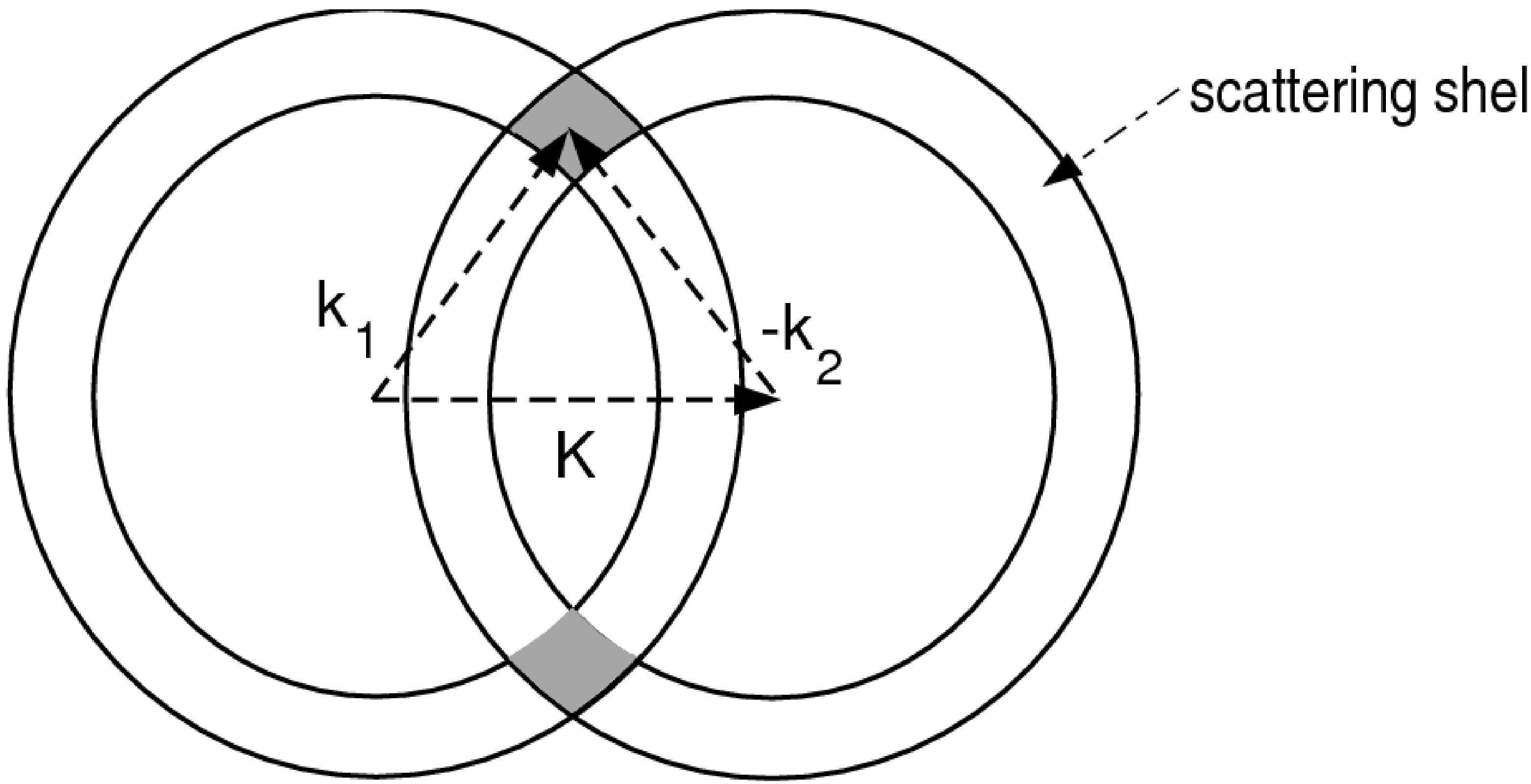
Scattering of Cooper Pairs



$E_k \sim k^2$



$$k_1 + k_2 = k_1' + k_2' = K$$



If the pair has a finite center of mass momentum, so that $k_1+k_2=K$, then there are few states which it can scatter into through the exchange of a phonon.

The Cooper Instability of the Fermi Sea

Now consider these two electrons above the Fermi surface. They will obey the Schroedinger equation.

$$-\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2)\psi(\mathsf{r}_1\,\mathsf{r}_2) + V(\mathsf{r}_1\,\mathsf{r}_2)\psi(\mathsf{r}_1\,\mathsf{r}_2) = (\epsilon\;\; + 2E_F)\psi(\mathsf{r}_1\,\mathsf{r}_2)$$

If V = 0, then ε = 0, and

$$\psi_{V=0} = \frac{1}{L^{3/2}}e^{i\mathsf{k}_1\cdot\mathsf{r}_1}\frac{1}{L^{3/2}}e^{i\mathsf{k}_2\cdot\mathsf{r}_2} = \frac{1}{L^3}e^{i\mathsf{k}(\mathsf{r}_1-\mathsf{r}_2)},$$

where we assume that $k_1 = -k_2 = k$. For small V, we will perturb around the V = 0 state, so that

$$\Rightarrow \quad \psi(\mathbf{r}_1 \mathbf{r}_2) = \frac{1}{L^3} \sum_{\mathbf{k}} g(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$$

The sum must be restricted so that

$$E_F < \frac{\hbar^2 \mathbf{k}^2}{2m} < E_F + \hbar\omega_D$$

this may be imposed by g(k), since $|g(k)|^2$ is the probability of finding an electron in a state k and the other in −k. Thus we take

$$g(\mathbf{k}) = 0 \quad \text{for} \quad \begin{cases} \mathbf{k} < \mathbf{k}_F \\ \mathbf{k} > \frac{\sqrt{2m(E_F + \hbar\omega_D)}}{\hbar} \end{cases}$$

The Schroedinger equations may be converted to a k-space equation by multiplying it by

$$\frac{1}{L^3}\int d^3\mathsf{r}\, e^{-i\mathsf{k}'\cdot\,\mathsf{r}} \Rightarrow \ \mathrm{S.E.}$$

so that

$$\frac{\hbar^2 k^2}{m} g(\mathsf{k}) + \frac{1}{L^3}\sum_{\mathsf{k}'} g(\mathsf{k}') V_{\mathsf{k}\mathsf{k}'} = (\epsilon\ + 2E_F) g(\mathsf{k})$$

where

$$V_{\mathsf{k}\mathsf{k}'} = \int V(\mathsf{r}) e^{-i(\mathsf{k}\ -\mathsf{k}')\cdot\mathsf{r}} d^3\mathsf{r}$$

$$V_{\mathsf{k}\mathsf{k}'} = \begin{cases} -V_0 & E_F < \frac{\hbar^2 \mathsf{k}^2}{2m}, \frac{\hbar^2 \mathsf{k}'^2}{2m} < E_F + \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \left(-\frac{\hbar^2 \mathsf{k}^2}{m} + \epsilon + 2E_F\right) g(\mathsf{k}) = -\frac{V_0}{L^3} \sum_{\mathsf{k}'} g(\mathsf{k}') \equiv -A$$

or

$$g(\mathsf{k}) = \frac{-A}{-\frac{\hbar^2 \mathsf{k}^2}{m} + \epsilon + 2E_F} \qquad \left(\text{i.e. for } E_F < \frac{\hbar^2 \mathsf{k}^2}{2m} < E_F + \hbar\omega_D\right)$$

Summing over **k**

$$\frac{V_0}{L^3} \sum_{\mathsf{k}} \frac{A}{\frac{\hbar^2 \mathsf{k}^2}{m} - \epsilon - 2E_F} = +A$$

or $$1 = \frac{V_0}{L^3} \sum_{\mathrm{k}} \frac{1}{\frac{\hbar^2 \mathrm{k}^2}{m} - \epsilon - 2E_F}$$

This may be converted to a density of states integral on E =$\hbar^2$k$^{2/}$2m

$$1 = V_0 \int_{E_F}^{E_F+\hbar\omega_D} Z(E_F) \frac{dE}{2E - \epsilon - 2E_F}$$

$$1 = \frac{1}{2} V_0 Z(E_F) \ln\left(\frac{\epsilon - 2\hbar\omega_D}{\epsilon}\right)$$

$$\epsilon = \frac{2\hbar\omega_D}{1 - e^{2/(V_0 Z(E_F))}} \simeq -2\hbar\omega_D e^{-2/(V_0 Z(E_F))} < 0, \qquad \text{as} \frac{V_0}{E_F} \to 0$$

The BCS Ground State

If w_k is the probability that a pair state ($k\uparrow$, $-k\downarrow$) is occupied then

$$E_{\text{kin}} = 2\sum_{\mathsf{k}} w_k \xi_k, \qquad \xi_k = \frac{\hbar^2 \mathsf{k}^2}{2m} - E_F$$

Annihilation and creation operators for the pair states labeled by k

$$|1\rangle_k \qquad (\mathsf{k}\uparrow, -\mathsf{k}\downarrow)\text{occupied}$$
$$|0\rangle_k \qquad (\mathsf{k}\uparrow, -\mathsf{k}\downarrow)\text{unoccupied}$$

$$\Rightarrow \quad |\psi_k\rangle = u_k |0\rangle_k + v_k |1\rangle_k$$

where $v_k^2 = w_k$ and $u_k^2 = 1 - w_k$

The BCS state may be written as

$$|\phi_{BCS}\rangle \simeq \prod_k \{u_k |0\rangle_k + v_k |1\rangle_k\}$$

By the Pauli principle, the state (k ↑, −k ↓) can be, at most, singly occupied, thus a (s = 1) Pauli representation is possible

$$|1\rangle_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_k \qquad |0\rangle_k = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_k$$

Where σ^+_k and σ^-_k, describe the creation and anhialation of the state (k ↑, −k ↓)

$$\sigma_k^+ = \tfrac{1}{2}(\sigma_k^1 + i\sigma_k^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_k^- = \tfrac{1}{2}(\sigma_k^1 - i\sigma_k^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Of course $\sigma_k^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}_k = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\sigma_k^+ \left|1\right\rangle_k = 0 \qquad \sigma_k^+ \left|0\right\rangle_k = \left|1\right\rangle_k$$
$$\sigma_k^- \quad \left|1\right\rangle_k = \left|0\right\rangle_k \qquad \sigma_k^+ \left|0\right\rangle_k = 0$$

The reduction of the potential energy is given by $<\phi_{BCS}|V|\phi_{BCS}>$

$$\left\langle \phi_{BCS} \left| V \right| \phi_{BCS} \right\rangle = -\frac{V_0}{L^3} \left\{ \prod_p \left(u_p \left\langle 0\right| + v_p \left\langle 1 \right|\right) \sum_{\mathsf{kk'}} \sigma^+_{\mathsf{k}} \sigma^-_{\mathsf{k'}} \right.$$
$$\left. \prod_{p'} \left(u_{p'} \left|0\right\rangle_{p'} + v_{p'} \left|1\right\rangle_{p'} \right) \right\}$$

Then as ${}_k\left\langle 1|1\right\rangle_{k'} = \delta_{kk'}$, ${}_k\left\langle 0|0\right\rangle_{k'} = \delta_{kk'}$ and ${}_k\left\langle 0|1\right\rangle_{k'} = 0$

$$\left\langle \phi_{BCS} \left| V \right| \phi_{BCS} \right\rangle = -\frac{V_0}{L^3} \sum_{kk'} v_k u_{k'} u_k v_{k'}$$

The total energy (kinetic plus potential) of the system of Cooper pairs is

$$W_{BCS} = 2\sum_k v_k^2 \xi_k - \frac{V_0}{L^3}\sum_{kk'} v_k u_{k'} u_k v_{k'}$$

Since $w_k = v_k^2$ and $1 - w_k = u_k^2$, we may impose this constraint by choosing

$$v_k = \cos\theta_k, \qquad u_k = \sin\theta_k$$

At T = 0, we require W_{BCS} to be a minimum.

$$\begin{aligned} W_{BCS} &= \textstyle\sum_k 2\xi_k \cos^2\theta_k - \frac{V_0}{L^3}\sum_{kk'} \cos\theta_k \sin\theta_{k'} \cos\theta_{k'} \sin\theta_k \\ &= \textstyle\sum_k 2\xi_k \cos^2\theta_k - \frac{V_0}{L^3}\sum_{kk'} \frac{1}{4}\sin 2\theta_k \sin 2\theta_{k'} \end{aligned}$$

$$\frac{\partial W_{BCS}}{\partial \theta_k} = 0 = -4\xi_k \cos\theta_k \sin\theta_k - \frac{V_0}{L^3}\sum_{k'} \cos 2\theta_k \sin 2\theta_{k'}$$

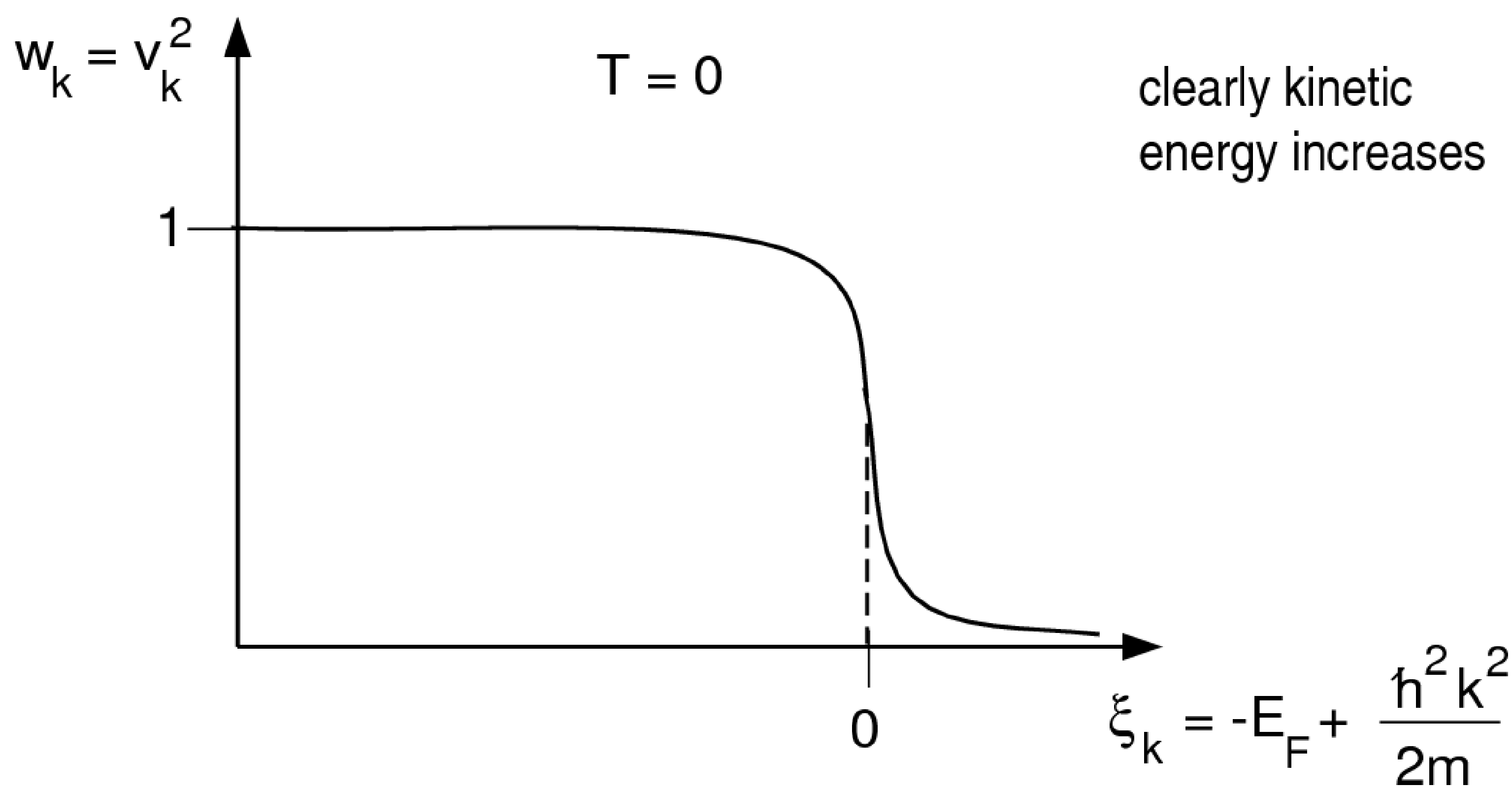
$$\xi_k \tan 2\theta_k = -\frac{1}{2}\frac{V_0}{L^3}\sum_{k'} \sin 2\theta_{k'}$$

Conventionally, one introduces the parameters $E_k = \sqrt{\xi_k^2 + \Delta^2}$, $\Delta = \frac{V_0}{L^3} \sum_k u_k v_k = \frac{V_0}{L^3} \sum_k \cos\theta_k \sin\theta_k$. Then we get

$$\xi_k \tan 2\theta_k = -\Delta \Rightarrow 2u_k v_k = \sin 2\theta_k = \frac{\Delta}{E_k}$$

$$\cos 2\theta_k = \frac{-\xi_k}{E_k} = \cos^2\theta_k - \sin^2\theta_k = v_k^2 - u_k^2 = 2v_k^2 - 1$$

$$w_k = v_k^2 = \frac{1}{2}\left(1 - \frac{-\xi_k}{E_k}\right) = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}\right)$$



If we now make these substitutions $\left(2u_k v_k = \frac{\Delta}{E_k}, v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right)\right)$ W_{BCS}, then we get

$$W_{BCS} = \sum_k \xi_k \left(1 - \frac{\xi_k}{E_k}\right) - \frac{L^3}{V_0}\Delta^2.$$

Compare this to the normal state energy, again measured relative to E_F

$$W_n = \sum_{k<k_F} 2\xi_k$$

or

$$\begin{aligned} \frac{W_{BCS} - W_n}{L^3} &= -\frac{1}{L^3}\sum_k \xi_k \left(1 + \frac{\xi_k}{E_k}\right) - \frac{\Delta^2}{V_0} \\ &\approx -\frac{1}{2} Z(E_F)\Delta^2 < 0. \end{aligned}$$